Roseville College

YEAR 12 Half Yearly Examination

2000

MATHEMATICS

2 Unit

2/3 Unit (Common)

Time Allowed - 3 hours + 5 minutes reading time

Instructions:

- Attempt all questions
- All questions are of equal value
- Begin each question on a new page, with your number clearly written at the top of EVERY page.
- Write on one side only, use blue or black pen.
- In every question. all necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Standard Integrals are provided.

QUESTION 1 - START A NEW PAGE

(a) Evaluate $\dfrac{\sqrt{410.2} \times 0.783}{4.926 + 5.604}$ correct to 3 significant figures

(2)

(b) Find the exact value of x
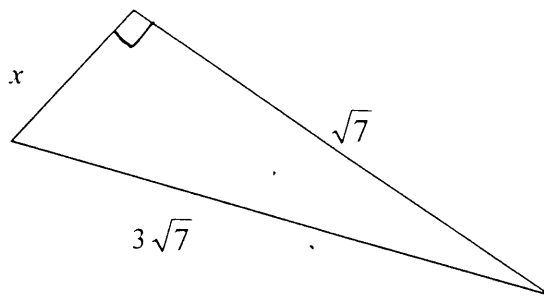
(2)

(c) Factorise $125 - x^3$

(1)

(d) Given

$$f(x) = \begin{cases} -5 & \text{for } x \le -3 \\ 2x & \text{for } -3 < x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$$

find the value of $f(-3) + f(3)$

(2)

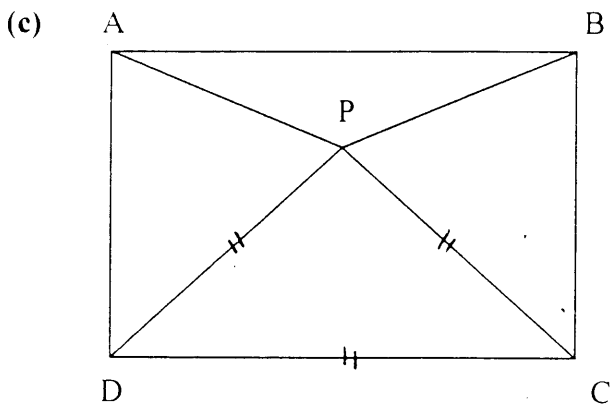**(e)** Graph the solution of the equation on a number line

$$|\, 3 - 2x \,| \le 10$$

(3)

(f) Simplify $\dfrac{1}{p-q} - \dfrac{1}{p}$

(2)

Marks

(a) A triangle has sides of length 4, 5, 7 units.
Find, to the nearest degree, the size of the largest angle. (2)

(b) Given $\sin \theta = \dfrac{8}{17}$ and θ is obtuse, find an exact value of $\cos \theta$ (2)

(c) A   B

P is a point inside a
square ABCD such that
PCD is equilateral.
Prove that $\triangle APD \equiv \triangle BPC$

D   C

Diagram not to scale (3)

(d) Find the equation of the tangent to the curve $y = x^2 - 4x$ at the point
where $x = 3$ (3)

**(e)** Express $0.0\dot{4}$ as a simplified fraction (2)

QUESTION 3 - START A NEW PAGE

(a) The points A and B have co-ordinates (4, 3) and (−2, 1) respectively.

    i)   Find the equation of AB

    ii) K is the point of intersection of the line AB and the y axis. Find the co-ordinates of K.

    iii) Calculate the perpendicular distance of point L (2,7) from AB. Leave your answer in exact form.

    iv) Hence, or otherwise, find the area of triangle ABL

(6)

(b)
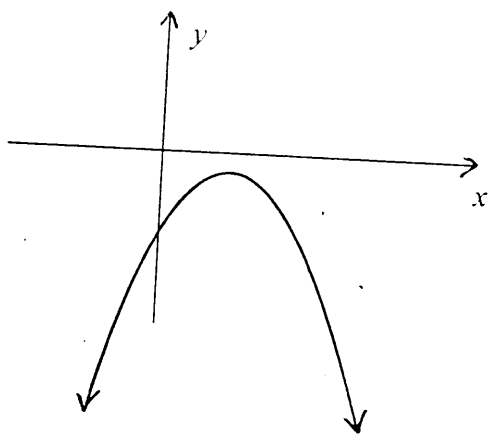
The graph shows the parabola $y = ax^2 - bx + c$
Comment on the sign of

(i) a

(ii) c

(iii)   the discriminant Δ

(3)

(c) For the equation $x^2 - (k + 1)x - 2(k + 1) = 0$ find the

    i)   discriminant in terms of k

   ii) values of k for which this equation has real roots

(3)

QUESTION 4 - START A NEW PAGE

(a)   i)   Sketch the graph of the parabola $x^2 = 4y$, showing the co-ordinates of the focus and the equation of the directrix.   (3)

ii)   Show that P(4,4), lies on the parabola and is equidistant from the focus and directrix.   (2)

(b)   Find the value of $\displaystyle\sum_{n=1}^{40} (2n+1)$ (2)

(c)   The two perpendicular lines

$$3x + 2y = 16$$
$$\text{and}\quad 2x + ay = b$$

intersect at the point (4,2). Find the values of a and b.   (2)

(d)
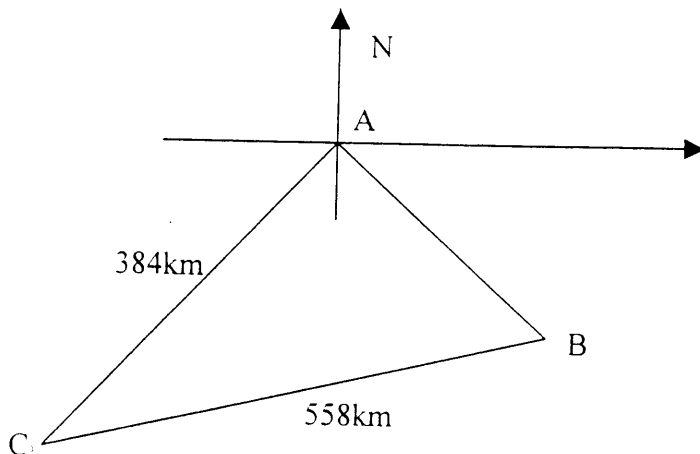
The diagram above shows the path of a flying saucer on a bearing of 235°T flying from colony A to colony C which is 384km away. At colony C, the saucer changes course and travels 558km on a bearing of 068°T to colony B   (3)

i)   Copy this diagram onto your answer sheet and show that $\angle$ ACB = 13°

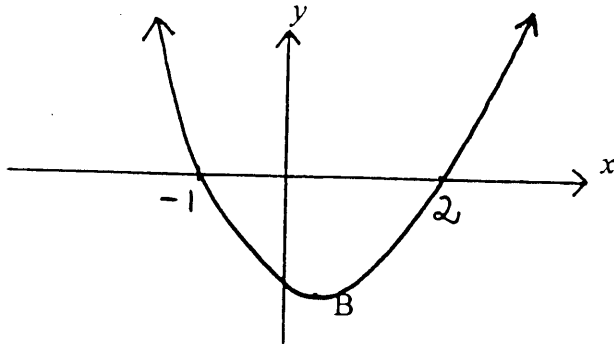ii)   Find the distance of colony B from colony A, giving your answer correct to the nearest kilometre.

QUESTION 5 - START A NEW PAGE
Mark

(a) The diagram represents the graph of the function (3)

$$f(x) = (x-p)(x-q)$$

(i) Write down the values of p and q

(ii) The function has 1 stationary point at B
Find the co-ordinates and nature of point B.

(b) Differentiate with respect to x (6)

(i) $\sqrt{x^3 - 1}$

(ii) $\dfrac{x+7}{x+4}$
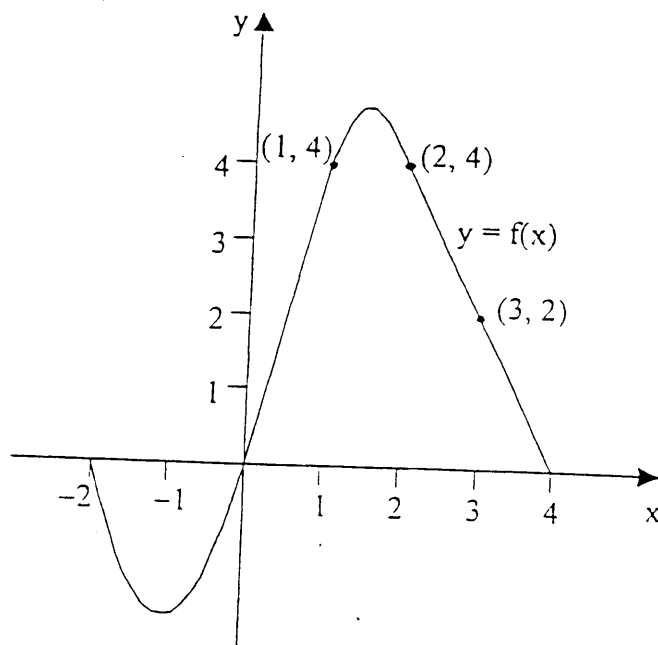
(iii) $x^4(3x-1)^3$

(c) Find (3)

(i) $\int 2\sqrt{x}\ dx$

(ii) $\int (2x-3)^4 dx$

QUESTION 6 - START A NEW PAGE

(a)

The curve $y = f(x)$ is shown above.

(i) Write down an expression for the exact area bounded by the curve and the x axis, between the limits of $x = -2$ and $x = 4$ (you are not required to find the area). (1)

(ii) Use Simpson's Rule with 5 function values to approximate the area enclosed by the curve, the x axis and the line $x = 0$ and $x = 4$. (3)

(b) (i) Draw a neat sketch of the curve $y = 4\cos 2x$ over the domain $0 \le x \le 2\pi$

(ii) What is the period of this graph?

(iii) What is the range of this function?

(iv) For what values of x will $4\cos 2x = 4$ (5)

(c) Find the exact value of $\displaystyle\int_{0}^{\frac{\pi}{2}} 3\sin\frac{x}{2}dx$ (3)
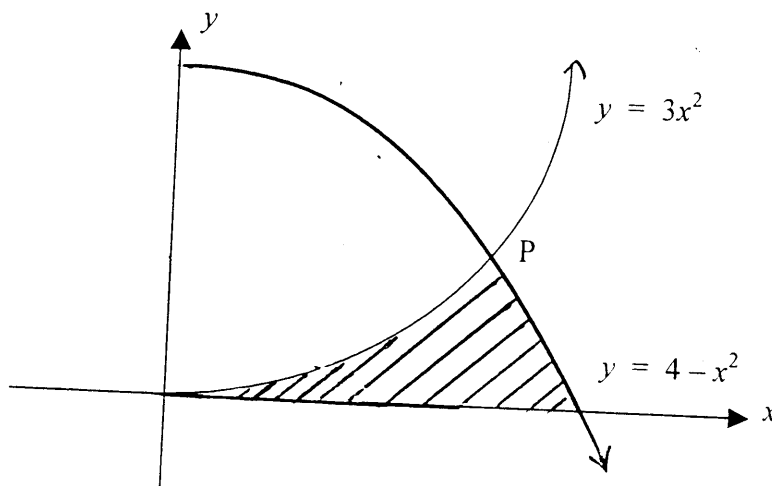
QUESTION 7 - START A NEW PAGE

(a)
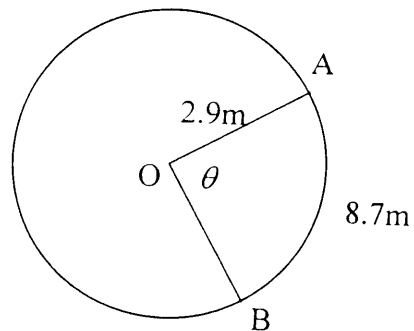
The radius is 2.9m
and arc AB has
length 8.7m

(i) Find the exact value of θ in radians

(2)

(ii) Find the area of the sector AOB

(2)

(b)

$y = 3x^2$

$y = 4 - x^2$

(i) Show that the co-ordinates of P are (1.3)

(ii) Find the value of the shaded area

(4)

(c) Simplify $\dfrac{\tan\theta \sec\theta}{1 + \tan^2\theta}$

(2)

(d) Solve $\sin^2\theta = \dfrac{3}{4}$ for $0 \le \theta \le 2\pi$

(2)

(a)   For a given arithmetic series $T_2 = -7$ and $T_9 = 28$

(i) Calculate the value of the first term and the common difference.

(ii) How many terms of this series must be added for a sum of 1313?   (5)

(b)   In its first year of operation, the "four dollar" shop sold 1000 sets of Pokemon cards. In its second year they sold 1300 sets of cards and its level of sales continued to increase by 30% above that of the previous year.

(i) How many sets of cards will it sell in its third year of operation?

(ii) If this trend continues, how many sets of cards can you expect the "four dollar" shop to sell over 15 years?   (3)

(c)   The first three terms of a geometric series are
$$(4 + x) + (1 + x) + (-\tfrac{1}{2} + x) + \ldots\ldots\ldots\ldots\ldots\ldots$$

(i) Calculate the value of x (2)

(ii) Does this series have a limiting sum?
Justify your answer.   (2)
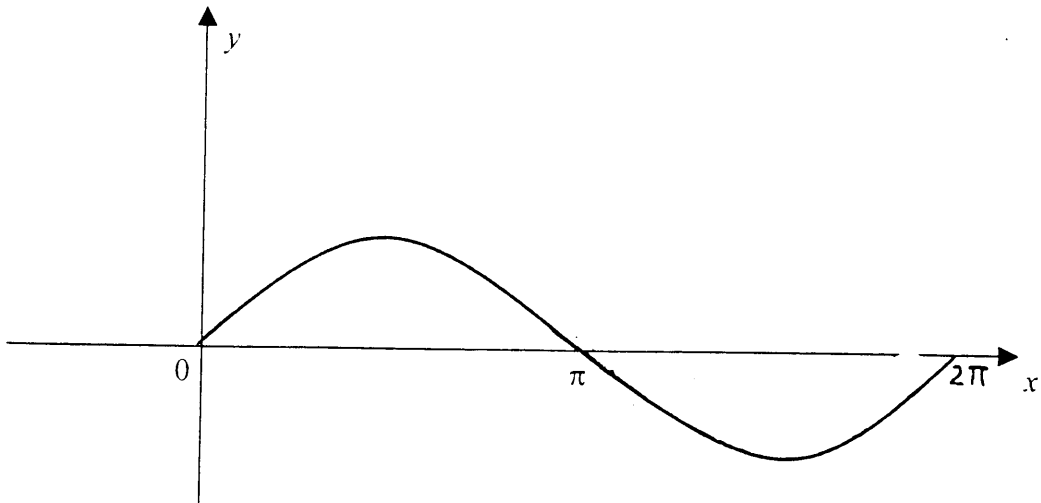
(a) The graph shows $y = \sin x$ for $0 \le x \le 2\pi$

(i) Find the exact area bounded by the curve $y = \sin x$ and the x axis (3)

(ii) Without calculation comment on the value of $\displaystyle\int_0^{2\pi} \sin x \, dx$ (1)

(b) (i) Poppy Star decides to save some money for her future. She has decided to invest \$3,000 at the beginning of each year for the next 40 years. If the interest rate is kept constant at 4½% p.a. (compound) how much can Poppy expect to have accumulated at the end of 40 years? (4)

(ii) If Poppy's investment fund allows her to withdraw her money at any time, calculate the amount at the beginning of the 20th year. (2)

(c) In the diagram AB = AC
BD bisects $\angle$ B
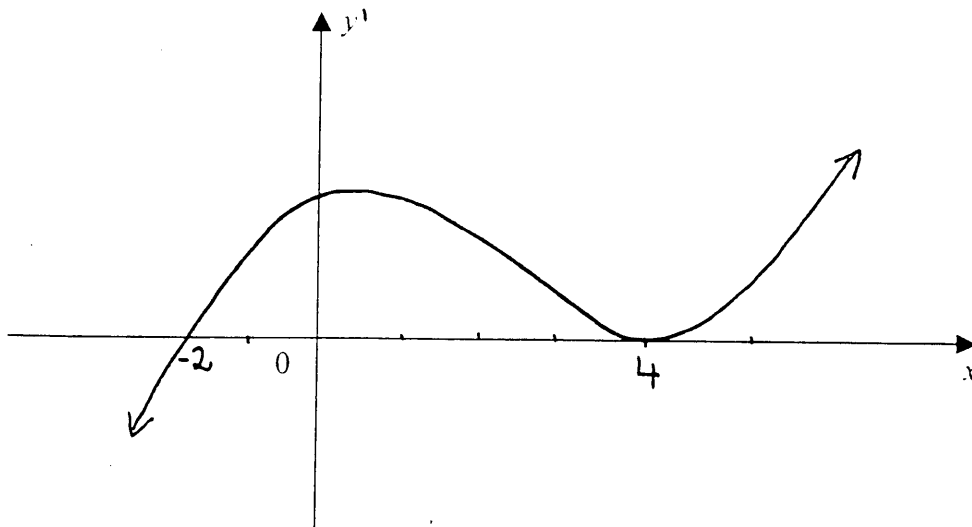$\angle$ B = 2 $\angle$A
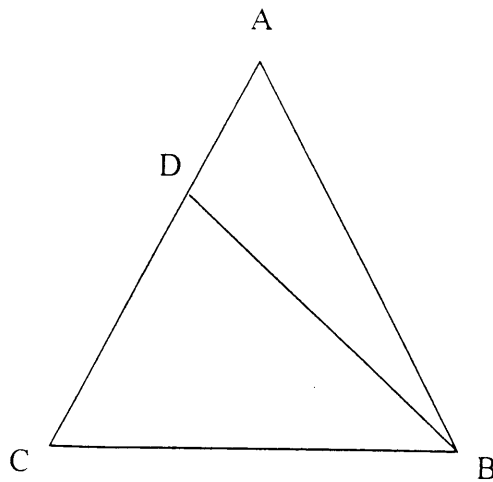
Find $\angle$ DBC (2)

QUESTION 10 - START A NEW PAGE

(a)   The diagram shows the <u>derivative</u> of $y = f(x)$

(i) Write down the x co-ordinate of the turning point on $y = f(x)$
and state whether it is a relative maximum or minimum.   (1)

(ii) At what value on $y = f(x)$ is there a horizontal point of inflection?   (1)

(iii) Where is the function $y = f(x)$ increasing?   (1)

(iv) Sketch a possible graph of $y = f(x)$.   (1)

(b)   (i) Differentiate the function

$$f(x) = 150\pi x - \frac{\pi x^3}{2}$$ (1)

(c)   A cylindrical perfume container, NOT including the top, is to be
made from a 300 π mm^2 sheet of expensive metal.

(i) Show an expression for the volume (V) of the can if
$V = 150\pi r - \dfrac{\pi r^3}{2}$ where r is the radius of the base.   (3)

(ii) Find the value of r which will allow for a maximum volume.   (2)

(iii) If the cost of the perfume and spray top is $15, and the cost of the
metal is 4c per mm^2, find the r.r.p. (recommended retail price) if
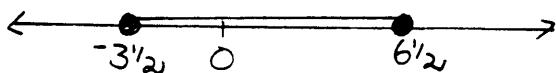the perfume is to make a 40% profit.   (2)

Question (1)

(a) $1.506021663 \div 1.51$ ②

(b) $x^2 = (3\sqrt{7})^2 - (\sqrt{7})^2$
$x^2 = 56$
$x = \sqrt{56} \ (2\sqrt{14})$ ②

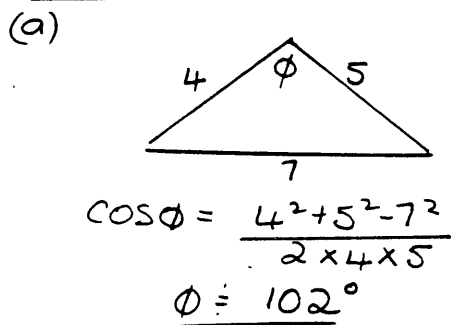(c) $(5-x)(25+5x+\overset{x^2}{\cancel{125}})$ ①

(d) $f(-3) + f(3)$
$= -5 + 3^2$
$= 4$ ②

(e) $3-2x \le 10$ $3-2x \ge -10$
$-2x \le 7$ $-2x \ge -13$ ⑤
$x \ge -3\frac{1}{2}$ $x \le 6\frac{1}{2}$

$-3\frac{1}{2}$ 0 $6\frac{1}{2}$

(f) $\dfrac{1}{p-q} - \dfrac{1}{p}$

$\dfrac{p - (p-q)}{p(p-q)}$

$= \dfrac{p-p+q}{p(p-q)}$ ②

$= \dfrac{q}{p(p-q)}$

Question (2).

(a)

$\cos\phi = \dfrac{4^2+5^2-7^2}{2\times4\times5}$

$\phi \doteq 102°$ ②

(b) $\cos\theta = \dfrac{-15}{17}$

  ②

(c) In $\triangle$S APD and BPC
$AD = BC$ (Equal sides of square)
$PD = PC$ (equal sides of $\triangle PCD$)
$\angle ADP = \angle PCB \ (90-60) = 30°$
$\triangle APD \equiv \triangle BPC \ (SAS)$ ③

(d) $y = x^2 - 4x$ at $(3, -3)$
$y' = 2x - 4$
when $x = 3$
$y' = 2 \Rightarrow m = 2$ ③
$\therefore y + 3 = 2(x-3)$
$y = 2x - 9$

(e) let $n = 0.04\dot{4}$
$10n = 0.444\cdots$
$100n = 4.444\cdots$
$\therefore 90n = 4$ ②
$n = \dfrac{2}{45}$

Question (3)

(a) $A(4,3)$ $B(-2,1)$.

(i) $m = \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore y - 1 = \dfrac{1}{3}(x+2)$.

$3y - 3 = x + 2$ ②

$x - 3y + 5 = 0$.

(ii) when $x = 0$
$y = \dfrac{5}{3}$ ①

(iii) $(2,7) \rightarrow x - 3y + 5 = 0$

$d = \dfrac{|2 - 21 + 5|}{\sqrt{1+9}}$

$= \dfrac{14}{\sqrt{10}} \times \dfrac{\sqrt{10}}{\sqrt{10}}$

$= \dfrac{7\sqrt{10}}{5}$ ①

(iv) $AB = \sqrt{(3-1)^2 + (4+2)^2}$
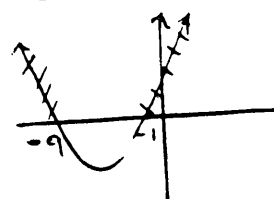$= \sqrt{4 + 36}$
$= \sqrt{40} \ (2\sqrt{10})$

$A = \dfrac{1}{2} \times \sqrt{40} \times \dfrac{7\sqrt{10}}{5}$

$A = 14 U^2$ ②

(b) (i) $a < 0$ ①
(ii) $c < 0$ ①
(iii) $\Delta < 0$ ①

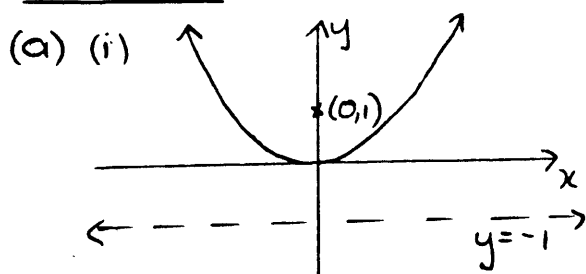(c) (i) $\Delta = b^2 - 4ac$
$= [-(k+1)]^2 + 8(k+1)$.
$= k^2 + 2k + 1 + 8k + 8$
$= k^2 + 10k + 9$ ①

(ii) Real Roots when $\Delta \ge 0$
$k^2 + 10k + 9 \ge 0$
$(k+9)(k+1) \ge 0$ ②

$\underline{k \le -9, \ k \ge -1}$

Question (4)

(a) (i)

$x^2 = 4y$
$\therefore a = 1$

③

(ii) Sub m $(4,4)$

$\therefore 4^2 = 4 \times 4$

LHS = RHS

$\therefore (4,4)$ lies on $x^2 = 4y$

$d_1 =$ distance $(4,4)$ $(0,1)$.

$d_1^2 = 4^2 + 3^2$

$= 25$

$\therefore d_1 = 5$

②

$d_2 =$ distance $y + 1 = 0$ and $(4,4)$

$d_2 = \dfrac{|0 + 4 + 1|}{\sqrt{0+1}}$

$= 5$

$\therefore (4,4)$ is equidistant from focus & directrix.

(b) $\displaystyle\sum_{n=1}^{40} (2n+1) = 3 + 5 + 7 + \cdots + 81$

A.P. with $a = 3, d = 2, n = 40$.

$\therefore S_{40} = \dfrac{40}{2}(3 + 81)$

②

$\underline{S_{40} = 1680}$

(c)

$3x + 2y = 16 \quad \cdots ①$

$2x + ay = b \quad \cdots ②$

sub in $(4,2)$.

$\therefore \quad 12 + 4 = 16$.

$8 + 2a = b \quad \cdots ③$

Since lines are $\perp$

$m_1 = -\dfrac{1}{m_2}$.

$\therefore -\dfrac{3}{2} = -\left(-\dfrac{a}{2}\right)$.

$-\dfrac{3}{2} = \dfrac{a}{2}$

$\therefore a = -3 \quad \rightarrow$ sub in ③

$8 - 6 = b$

$b = 2 \qquad \underline{a = -3, \ b = 2}$

(d)
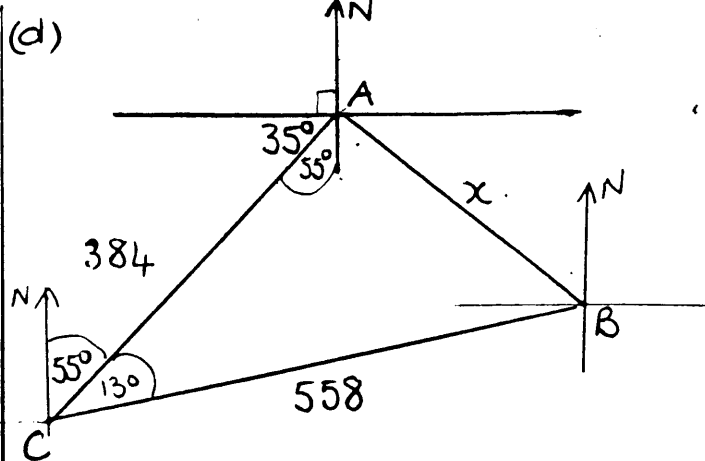
(i) $\angle ACB = 55° \ (235° - 180°)$.

$\therefore \angle NCA = 55°$ (alt $\angle$'s, CN $\parallel$ AN)

$\angle ACB = 68 - 55$

$\underline{\angle ACB = 13°}$ ①

(ii) $x^2 = (384)^2 + (558)^2 - 2(384)(558)\cos 13°$

$\underline{x \doteqdot 203\,km}$ ②

Question (5)

(a) $f(x) = (x-p)(x-q)$ ①

(i) $\underline{p = -1, \ q = 2}$ or $\underline{p = 2, q = -1}$.

(ii) at B $\quad x = \frac{1}{2} \quad \rightarrow$ MIN.

$\therefore f(\frac{1}{2}) = -1$

$\therefore \underline{\text{min point at } (\frac{1}{2}, -1)}$. ②

(b) (i) $y = (x^3 - 1)^{1/2}$

$y' = \frac{1}{2}(x^3 - 1)^{-1/2} \cdot 3x^2$

$= \dfrac{3x^2}{2\sqrt{x^3 - 1}}$ ②

(ii) $y = \dfrac{x+7}{x+4}$

$y' = \dfrac{1(x+4) - 1(x+7)}{(x+4)^2}$

$= \dfrac{x+4-x-7}{(x+4)^2}$

$= \dfrac{-3}{(x+4)^2}$ ②

(iii) $y = x^4(3x-1)^3$

$y' = 4x^3(3x-1)^3 + 3(3x-1)^2 \cdot 3 \cdot x^4$

$= 4x^3(3x-1)^3 + 9x^4(3x-1)^2$

$= x^3(3x-1)^2[21x - 4]$ ②

c) (i) $\int 2\sqrt{x}\, dx$

$= \int 2x^{1/2}\, dx$

$= 2x^{3/2} \cdot \dfrac{2}{3} + c$

$= \dfrac{4}{3} x^{3/2} + c$ $\quad$ ①½

(ii) $\int (2x-3)^4\, dx$

$= \dfrac{(2x-3)^5}{10} + c$ $\quad$ ①½

Question (6)

(i) $A = \left| \int_{-2}^{0} f(x)\, dx \right| + \int_{0}^{4} f(x)\, dx$ $\quad$ ①

(ii)

$\overbrace{}^{h=1}$

x	0	1	2	3	4
f(x)	0	4	4	2	0

$A \doteqdot \dfrac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4) \right]$

$\doteqdot \dfrac{1}{3}\left[0 + 16 + 8 + 8 + 0 \right]$ $\quad$ ③

$A \doteqdot 10\,^{2}/_{3}\, U^2$

(b) (i) $y = 4\cos 2x$ $\qquad a = 4$

$\qquad\qquad\qquad\qquad\qquad$ period $= \pi$

②
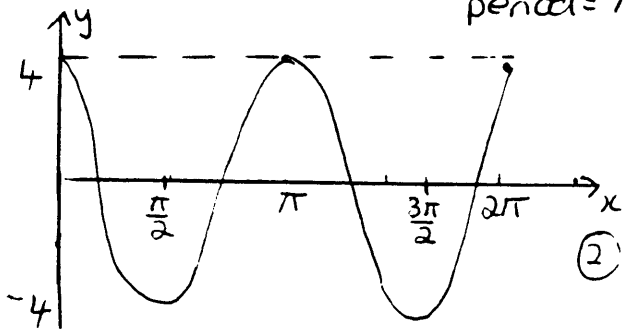
(ii) period $= \pi$ $\quad$ ①

(iii) $-4 \le y \le 4$ $\quad$ ①

(iv) $x = 0, \pi, 2\pi$ $\quad$ ①

(c) $\int_{0}^{\pi/2} 3\sin \dfrac{x}{2}\, dx$

$\left[-6\cos \dfrac{x}{2} \right]_{0}^{\pi/2}$

$= \left(-6\cos \dfrac{\pi}{4} \right) - \left(-6 \cdot \cos 0 \right)$

$= -\dfrac{6}{\sqrt{2}} + 6$

$= 6 - \dfrac{6}{\sqrt{2}}$ or $(6 - 3\sqrt{2})$. $\quad$ ③

Question (7)

(a)

(i) $r\theta = 8.7$

$2.9\theta = 8.7$

$\theta = 3$ $\quad$ ②

(ii) $A = \dfrac{1}{2} r^2 \theta$

$= \dfrac{1}{2} \times 2.9 \times 2.9 \times 3$

$A = 12.615\, m^2$ $\quad$ ②

(b) $y = 3x^2$

$y = 4 - x^2$

$\therefore \ 3x^2 = 4 - x^2$

$4x^2 - 4 = 0$

$x^2 - 1 = 0$

$x = \pm 1$. but $x \ne -1$. $\quad$ ②

when $x = 1$, $y = 3$ $\therefore$ P is $(1, 3)$

(ii) $A = \int_{0}^{1} 3x^2\, dx + \int_{1}^{2} 4 - x^2\, dx$

$= \left[x^3 \right]_{0}^{1} + \left[4x - \dfrac{x^3}{3} \right]_{1}^{2}$

$= [1 - 0] + \left[\left(8 - \dfrac{8}{3} \right) - \left(4 - \dfrac{1}{3} \right) \right]$

$= 1 + 1\,^{2}/_{3}$ $\quad$ ③

$= 2\,^{2}/_{3}\, U^2$

(c) $\dfrac{\tan\theta \sec\theta}{1 + \tan^2\theta}$

$= \dfrac{\tan\theta \sec\theta}{\sec^2\theta}$

$= \dfrac{\tan\theta}{\sec\theta}$

$= \dfrac{\sin\theta}{\cos\theta} \times \dfrac{\cos\theta}{1}$

$= \sin\theta$ $\quad$ ②

Question (8)

(a) $T_2 = a + d = -7$ .... ①
$T_9 = a + 8d = 28$ .... ②

② - ①
$7d = 35$
$d = 5$

Sub into ①
$a + 5 = -7$
$\therefore a = -12$

$\therefore \underline{a = -12 \text{ and } d = 5}$ ②.

(ii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$1313 = \frac{n}{2}[-24 + (n-1)5]$

$2626 = n[5n - 29]$
$2626 = 5n^2 - 29n$

$\therefore 5n^2 - 29n - 2626 = 0$

$n = \frac{29 \pm \sqrt{53361}}{10}$ ③.

$= 26, -20.2$
↓ invalid

$\therefore \underline{n = 26}$

(b) 1000, 1300, 1690

(i) 1690 sets of cards ①

(ii) $S_{15} = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1000(1.3^{15} - 1)}{0.3}$ ②.

$= 167286 \cdot 31$

$\doteq \underline{167286 \text{ sets of cards}}$

(c) (i) $\frac{1+x}{4+x} = \frac{-\frac{1}{2} + x}{1 + x}$

$(1+x)^2 = (x - \frac{1}{2})(4 + x)$

$1 + 2x + x^2 = 4x + x^2 - 2 - \frac{1}{2}x$

$-\frac{3}{2}x = -3$

$\underline{x = 2}$ ②

(d) limiting sum when $|r| <$

$\left|\frac{1+x}{4+x}\right| < 1$.

for $x = 2$

$\left|\frac{3}{6}\right| < 1$.

$\therefore$ it does have a limiting sum.

Question (9)

(a) (i) $A = 2\int_0^\pi \sin x \, dx$.

$= 2\left[-\cos x\right]_0^\pi$

$= 2\left[-\cos\pi - (-\cos 0)\right]$

$= 2\left[1 + 1\right]$

$= 4u^2$

(ii) $\int_0^{2\pi} \sin x \, dx = 0$

(b) (i) $A_1 = 3000(1.045)^{40}$
$A_2 = 3000(1.045)^{39}$
$\vdots$
$A_{40} = 3000(1.045)^1$

Total $= 3000(1.045)^{40} + 3000(1.045)^{39}$
$+ \ldots \quad 3000(1.045)^1$

$= 3000(1.045)^1 + 3000(1.045)^2 + \ldots$
$+ 3000(1.045)^{40}$

$= 3000\left[1.045 + 1.045^2 + \ldots 1.045^{40}\right]$

G.P. with $a = 1.045$
$r = 1.045, n = 40$.
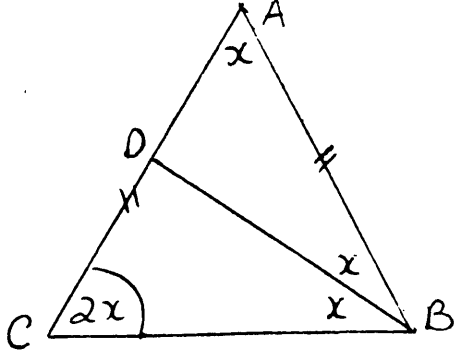
$= 3000\left[\frac{1.045(1.045^{40} - 1)}{0.045}\right]$ ④

$= \$335540 \cdot 06$

(ii) $A_{20} = 3000\left[\frac{1.045(1.045^{20} - 1)}{0.045}\right]$

$= \$94114 \cdot 27$ ②

$*(\text{or } \$97114 \cdot 27)$

.)

let $\angle ABC = 2x$

$\therefore \angle CAB = x$ (data)

$\angle ACB = 2x$ (equal $\angle$'s of ISOS$\triangle$)

$\therefore 5x = 180$ (angle sum $\triangle$)

$x = 36°$

$\therefore \angle OBC = 36°$
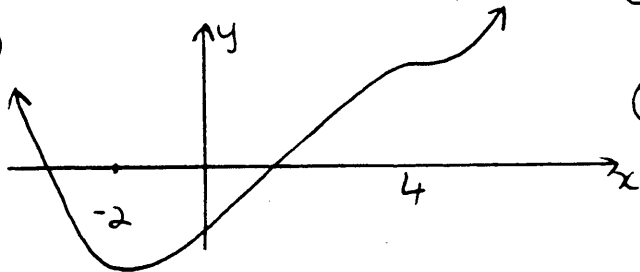
Question. (10)

(a)

(i) $x = -2 \rightarrow$ minimum ①

(ii) $x = 4$ ①

(iii) $-2 < x < 4$, $x > 4$ ①

(iv) ①

(b) (i) $f(x) = 150\pi x - \dfrac{\pi x^3}{2}$ ①

$f'(x) = 150\pi - \dfrac{3\pi x^2}{2}$

(c) (i) $\pi r^2 + 2\pi r h = 300\pi$

$r^2 + 2rh = 300$

$2rh = 300 - r^2$

$\therefore h = \dfrac{300}{2r} - \dfrac{r}{2}$

$h = \dfrac{150}{r} - \dfrac{r}{2}$

$V = \pi r^2 h$ ③

$= \pi \cdot r^2 \left(\dfrac{150}{r} - \dfrac{r}{2} \right)$

$= 150\pi r - \dfrac{\pi r^3}{2}$

(ii) $V' = 150\pi - \dfrac{3\pi r^2}{2}$

$V'' = -3\pi x$

max occurs at $V' = 0$.

$150\pi - \dfrac{3\pi r^2}{2} = 0$

$300\pi - 3\pi r^2 = 0$

$100 - r^2 = 0$

$r = \pm 10$ but $x \neq -10$. ②

check for max

$V'' = -30\pi < 0 \therefore \wedge$ max.

Max volume when $\underline{r = 10mm}$.

(iii) $(300\pi \times 0.04) + 15 = $ cost.

r.r.p. $= 1.4 [12\pi + 15]$ ②

$\underline{r.r.p = \$73.78}$